AUS | American University of Sharjah

**Score:**

**American University of Sharjah**
**Department of Mathematics and Statistics**

**Final Exam - spring 2018**
**MTH 213 – Discrete Math**

Each got 2 to 3 Extra

**Instructor Name:** **Ayman Badawi**

**Student Name:** Reem Elfatih Salman
**Student ID Number:** 71008

1. *No Questions are allowed during the examination.*
2. *This exam has 6 pages plus this cover page .*
3. *Do not separate the pages of the exam.*
4. *Scientific calculator are allowed but cannot be shared. Graphing Calculators are not allowed.*
5. *Take off your cap. Turn off all cell phones and remove all headphones.*
6. *No communication of any kind is permitted.*
7. *All working must be shown*

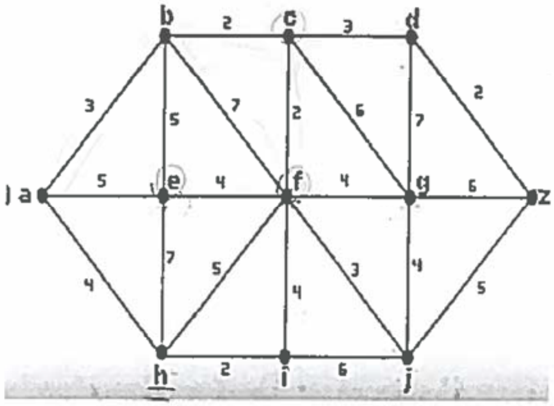
Student signature: ____________________

# Final Exam: MTH 213, Spring 2018

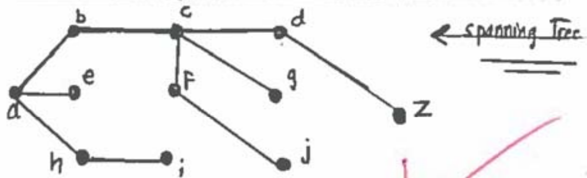
Ayman Badawi

**QUESTION 1. (6 points)**



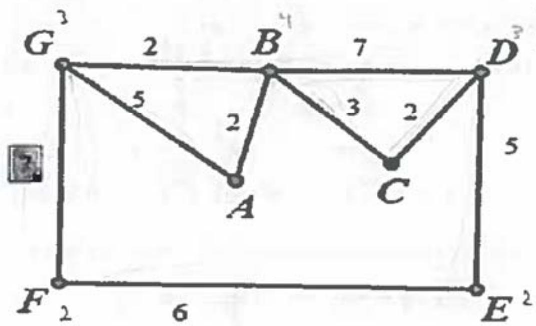
Use Dijkstar's method to find the minimum spanning tree (you may start from vertex a). Then draw such spanning tree

| | a | b | c | d | e | f | g | z | h | i | J |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | $\boxed{0a}$ | 3a | ∞ | ∞ | 5a | ∞ | ∞ | ∞ | 4a | ∞ | ∞ |
| b | | $\boxed{3a}$ | 5b | ∞ | 5a | 10b | ∞ | ∞ | 4a | ∞ | ∞ |
| c | | | $\boxed{5b}$ | 8c | 5a | 7c | 11c | ∞ | 4a | ∞ | ∞ |
| h | | | | 8c | 5a | 7c | 11c | ∞ | $\boxed{4a}$ | 6h | ∞ |
| e | | | | 8c | $\boxed{5a}$ | 7c | 11c | ∞ | " | 6h | ∞ |
| i | | | | 8c | | 7c | 11c | ∞ | | $\boxed{6h}$ | 12 |
| f | | | | 8c | | $\boxed{7c}$ | 11c | ∞ | | | 10 |
| d | | | | $\boxed{8c}$ | | | 11c | 10d | | | 10 |
| z | | | | | | | 11c | $\boxed{10d}$ | | | 10 |
| J | | | | | | | 11c | | | | $\boxed{10}$ |
| g | | | | | | | $\boxed{11c}$ | | | | |




← spanning Tree

**QUESTION 2. (6 points)**




1. A person wants to travel from vertex A and visits each other vertex EXACTLY once, then returns back to A. What is the shortest path (cycle) should he use? (SHOW THE WORK)

A – B – C – D – E – F – G

$2 + 3 + 2 + 5 + 6 + 7 + 5 = 30$

2. Is the graph Eulerian? If yes, then give me one Eulerian circuit. If no, then explain (briefly)

Not Eulerion because all degrees must be even (e.g deg (D) = 3)

3. Is the graph an Euler path? If yes, then give me such path. If no, then explain.

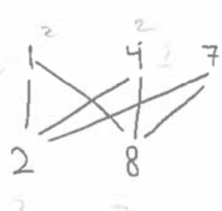
Yes, because exactly two degrees are odd

4. Is the graph a bi-partite graph? explain

No, there are odd cycles

**QUESTION 3. (6 points)** Let $D$ be a graph with vertex-set $V = \{1, 2, 4, 7, 8\}$. Two distinct vertices, say $a, b$, are connected by an edge iff $a + b = 3c$ for some $c \in N^*$. By drawing the graph, convince me that $D$ is a complete bi-partite graph. Is the graph Hamiltonian? if yes, then give me such cycle. If the graph is an Euler path, then give me such path.

↑ NOT Hamiltonian, no such cycle




Eular path because exactly two degrees are odd

path is ~~1 – 2 – 4 – 8 – 7 – 2~~

Euler path: 2-4-8-1-2-7-8

[redacted]

[redacted]

(iii) (**5 points**)Use math induction and prove that $12 \mid (8^{3n} - 20)$, for every integer $n \geq 1$.

(a) Prove for $n=1$

$5^{4(1)} - 1 = 624 \qquad 52 \times 12 = 624 \Rightarrow 12 \mid 624 \therefore$ True

(b) assume $12 \mid (8^{3n} - 20)$ is true for some $n \geq 1$

(c) Prove for $n+1$

$\therefore 12 \mid 5^{4n+4} - 1$

$\Rightarrow 5^{4n}5^{4} - 1 = 5^{4n}5^{4} - 5^{4} + 5^{4} - 1$

$= 5^{4}(5^{4n} - 1) + 5^{4} - 1$

divides by 12 (true by (b)) divides by 12 (true by (a))

$\Longrightarrow$ It is true

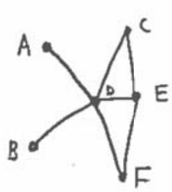
**QUESTION 5.** (**5 points**) Is the sequence 5, 3, 2, 2, 1, 1 graphical? If yes, then draw a graph with the given properties. If such graph exists, then it is possible that such graph has many "shapes". Hence without relying on the shape of such graph, convince me that such graph is NEVER a tree.

5, 3, 2, 2, 1, 1

$\Rightarrow$ 2, 1, 1, 0, 0

$\Rightarrow$ 0, 0, 0, 0 $\therefore$ such graph exists

Since there is always going to be a cycle in the graph, it can never be a tree




$|E| = \frac{\text{sum degrees}}{2} = 7$

$|V| = 6$

$|E| \neq |V| - 1 \Rightarrow$ NO Tree

**QUESTION 6.** (i) Let $A = \{0,1,2,3,4,5,6,7,8\}$ and $B = \{0,3,6\}$. Define "=" on A so that $a = b$ if $(a-b)\ mod(9) \in B$

a. **(4 points)** Show that "=" is an equivalence relation. (Hint: I guess that you need to show that $-r\ mod(9) \in B$ for every $r \in B$, also show that $(m+n)\ (mod\ 9) \in B$ for every $m, n \in B$)

(a) A-A: $(a-a) \bmod 9 = 0 \bmod 9 = 0 \in B$, always true

(b) A-B: assume $a=b$ so $a-b = k \pmod 9) \in B$

then $b-a = -k \pmod 9) = 9-(K \bmod 9) = \boxed{?}$

$K \in B$ so $K = \{0,3,6\}$ ↑

$k=3 \Rightarrow 9-(3 \bmod 9) = 6 \in B$

$k=6 \Rightarrow 9-(6 \bmod 9) = 3 \in B$ ∴ second axiom

$k=0 \Rightarrow 0 \bmod 9 = 0$ AlWAYS always holds

$9q+0 = a-b$

$9q+3 = a-b$

$9q+6 = a-b$

(c) A-B-C: $a=b$ and $b=c$ so $a \overset{?}{=} c$, this is a finite set so we can prove by example

b. **(3 points)** Find all equivalence classes of $A$.

$\bar{0} = \{0,3,6\}$

$\bar{1} = \{1,4,7\}$

$\bar{2} = \{2,5,8\}$

assume $a=6$ $b=3$ $c=0$

$a-b = 6-3 = 3 \bmod 9 = 3 \in B$

$b-c = 3-0 = 3 \bmod 9 = 3 \in B$

$a-c = 6-0 = 6 \bmod 9 = 6 \in B$

$\Rightarrow$ TRUE, similarily with

$a=1, b=4, c=7$

and $a=2, b=5, c=8$

$\frac{a-b-6}{9} = q$

$\frac{|a-b|-3}{9} = q$

c. **(1 points)** If we view " = " as a subset of $A \times A$, how many elements does "=" have? Do not list the elements, just tell me the cardinality of "=".

$3^2 + 3^2 + 3^2 = 27$

(ii) Let $A = \{-16, 1, 3, 10, 12, 15\}$ and $B = \{0, -3, -5, -9, -11, -12, -14, -19, -28, -31\}$. Let $a, b \in A$. Define "$\leq$" on A so that $a \leq b$ if $(b-a) \in B$. Then "$\leq$" is a partial relation on $A$ (DO NOT SHOW THAT)

a. **(4 points)** Draw the Hassee diagram of such relation


b. **(5 points)** By staring at the Hassee diagram, If possible, find

i. $12 \wedge 10$ 15

ii. $10 \wedge 3$ 15

iii. $1 \wedge 3$ 12

iv. $15 \vee -16$ −16

v. $1 \wedge -16$ 12

vi. $1 \vee 15$ 1

vii. $10 \vee 3$ 15

viii. ~~$10 \wedge -16$~~ 15

ix. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find $c$ no such C

x. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find $m$ $m = 15$

**QUESTION 7. (5 points)** For the following Algorithm segment. Find the exact number of additions, multiplications, and subtractions that will be performed when the algorithm is executed. Then find the complexity of the Algorithm segment. (SHOW THE WORK)

$m = 0; s = -20$

*For* $k := 1$ *to* $2n^3$

$L = k^2 + 3 - n - s^3 + 6$

*For* $i := 1$ *to* $\lceil \frac{k}{2} \rceil$

$s = 20 + m^3 + i^2 - k$

*next* $i$

*next* $k$

7

outer loop

$2n^3 - 1 + 1 = 2n^3$

$K=1 \Rightarrow 2$

$k=2 \Rightarrow 2$

Outer loop

$1 \quad \cdots \quad 2n^3$

$7 + 1(6) + 7 + 2(6) + 7 + 3(6) + 7 + 4(6) \cdots 7 + 2(\frac{n^3}{2})$

if $n$ = even

$\lceil \frac{k}{2} \rceil = \frac{k}{2}$

$(7 + 1(6)) + (7 + 2(6)) \ldots 7 + (\frac{k-2}{2})(6)$

$\frac{k}{2} - 1 = \boxed{\frac{k-2}{2}} = \frac{2n^3 - 2}{2}$

$\text{Complexity} = \frac{((7+(6)) + (7 + 6(\frac{2n^3-2}{2})) 2n^3}{2}$

$\boxed{O(C) = n^3 \cdot n^3 = n^6}$

if $n$ = odd

$\lceil \frac{k}{2} \rceil = \frac{k+1}{2}$

$(7 + 1(6)) + (7 + 2(6))$

$\cdots + 7 + (\frac{k-1}{2})(6)$

$\frac{k+1}{2} - 1 = \boxed{\frac{k+1-2}{2}} = \frac{2n^3 - 1}{2}$

$\text{Complexity} = \frac{2n^3((7 + 1(6)) + (7 + 6(\frac{2n^3-1}{2})))}{2}$

$O(C) = n^3 \cdot n^3 = n^6$

**QUESTION 8.** (i) **(2 points)** If possible solve $4x = 5$ over Planet $Z_{10}$.

$\gcd(4, 10) = 2 / 5$

$\uparrow$ 2 does not divide 5

$\Rightarrow$ no solution

(ii) **(3 points)** If possible solve $4x = 6$ over Planet $Z_{14}$. Then if possible solve $4x \equiv 6 \pmod{14}$ over Planet $Z$.

$\gcd(4, 14) = 2 \mid 6$ ✓

$\uparrow$ 2 solutions

$4x = 6 \pmod{14}$

$X = 5 \qquad X = 12$

$X = 5 + 14k, \ k \in Z \qquad X = 12 + 14m, \ m \in Z$

(iii) **(5 points)** Let $X$ be the number of students who will take MTH213 in the fall of 2018. Given $1 \leq X \leq 72$, $X \equiv 8 \pmod{9}$, and $X \equiv 3 \pmod{8}$. Use the Chinese Remainder Theorem (CRT) to find the value of $X$. (SHOW THE WORK)

$\gcd(9, 8) = 1 \Rightarrow$ CRT O.K!

$d_1 = (8)^{-1} \bmod 9 = 8$

$d_2 = (9)^{-1} \bmod 8 = 1$

$8x = 1 \pmod 9$

$x = 8$

$9x = 1 \pmod 8$

$\Rightarrow (8 \times 8 \times 8 + 3 \times 9 \times 1) \bmod 72$

$539 \bmod 72$

$= \boxed{35}$

(iv) (2 points) Find $(177)_8 \cdot (65)_8$

[redacted]

(vi) (3 points) It is clear that $gcd(14, 22) = 2$. Find two integers $a, b$ such that $22a + 14b = 2$ (SHOW THE WORK)

$2 = 8 - 6 \times 1$
$2 = 8 - (14 - 8 \times 1) \times 1$
$= 8 - 14 + 8 \times 1 = 8 \times 2 - 14$
$= (22 - 14 \times 1) \times 2 - 14 = 22 \times 2 - 14 \times 2 - 14 = 22 \times 2 - 14 \times 3$

$22 = 14 \times 1 + 8$, $\quad 14 = 8 \times 1 + 6$, $\quad 8 = 6 \times 1 + 2$, $\quad 6 = 2 \times 3 + 0$

$\boxed{a = 2 \quad b = -3}$

(vii) (2 points) Without solving the equation $10x^8 + 14x^7 - 21x + 28 = 0$, how can you conclude that it has no rational roots? (Short answer!)

[redacted]

(viii) (4 points) Given $a_n$ is a sequence such that $a_0 = 6$, and $a_1 = 12$, and $a_n = ua_{n-1} + va_{n-2}$ for every $n \geq 2$, where $u, v$ are some fixed real numbers; also given $a_n = a(-4)^n + bn(-4)^n$ for every $n \geq 0$, where $a, b$ are some fixed real numbers. Find the values of $a, b, u, v$.

$a_n = a(-4)^n + bn(-4)^n$
$a_0 = \boxed{a = 6}$
$a_1 = a(-4) + b(-4) = 6(-4) - 4b = 12$
$-24 - 4b = 12$
$-36 = -4b$
$\boxed{-9 = b}$

$a_n = u a_{n-1} + v a_{n-2}$
$a_2 = 12u + 6v = -192$
$a_3 = -192u + 12v = 1344$
$\boxed{u = -8}$ $\boxed{V = -16}$

$\Rightarrow a_2 = 6(-4)^2 + (-9)(2)(-4)^2 = -192$
$a_3 = 6(-4)^3 + (-9)(3)(-4)^3 = 1344$

(ix) Given $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $f$ be a bijective function from $S$ onto $S$ such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 4 & 5 & 6 & 9 & 7 & 2 & 1 & 3 \end{pmatrix}$$

a. (2 points) Find $f^2$ (i.e., find $f o f$). (Note that by staring at $f$, we understand that $f(1) = 8, \ldots, f(9) = 3$)

$F^2 = (1 \ 6 \ 9 \ 7 \ 3 \ 2 \ 4 \ 8 \ 5)$

b. (3 points) Find the least positive integer $n$ such that $f^n = I$, where $I$ is the identity map (i.e., $I(a) = a$ for every $a \in S$)

$(1, 8)\ (2, 4, 6, 7)\ (3, 5, 9)$

$LCM(2, 4, 3) = 12$

$LCM(2, 4) = \frac{2 \times 4}{gcd = 2} = 4$

$LCM(4, 3) = 12$

$\Rightarrow F^{12}$ is $I$

**QUESTION 9.** (i) **(2 points)** How many 4-digit odd numbers greater that 6000 can be formed using the digits (1, 2, 3, 4, 5, and 6).

$1C1 \times 6C1 \times 6C1 \times 3C1$

(ii) **(2 points)** There are 9 dots randomly placed on a circle such that exactly 6 of them are red and the remaining three dots are blue. How many triangles can be formed within the circle (i.e., inside the circle) such that each triangle has exactly two red vertices?

$6C2 \times 3C1$

5

(iii) **(2 points)** 2019 females are standing in a row and each female shakes hands only with the female on her left and with the female on her right exactly once. How many handshakes took place? (note that Female number 1 will receive only one shake hand and so is female number 2019)

$1 \times 2 \times 2 \times 2 \times 2 \ldots \times 1$

$1 \quad 2 \ldots 2018 \quad 2019$

$\underbrace{2018 - 2 + 1}$ $\Rightarrow 1 \times 2^{2017} \times 1$

(iv) **(2 points)** 202 persons are in a gathering, it is observed that more than 60% of them are females. We know that each person was born on Monday or Friday or Sunday or Tuesday. Then there exist at least $n$ persons who were born on the same day (i.e., Monday or Friday or Sunday or Tuesday). What is the maximum value of $n$?

$2017C_2$

$n = \left\lceil \frac{202}{4} \right\rceil = 51$

**QUESTION 11. (6 points)** Write DOWN T OR F

(i) If there exists a bijective function from $Q \cap [-4, 4]$ onto $N$, then there exists a bijection function from $Q \cap [-4, 4]$ onto $Q$ T

$S$

$|S| = |N| = |Q|$

(ii) $0101 \ominus 1001 = 1100$ T

(iii) $\forall x \in Q^*, \exists$ a nonzero $y \in Z$ such that $xy \in Z^*$. T

$x = 2$, $y = 1$; $x = \frac{1}{2}$, $y = 2$

(iv) It is a correct answer if one says that $gcd(4, 14) = -2$ over Planet Z. (Remember our secret!) T

(v) $\{3, \{3\}\} - \{3\} = \{3\}$ F $\{\{3\}\}$

(vi) If $\exists!\ x \in N^*$ such that $x^2 - 4 = 0$, then $\exists!\ y \in R^*$ such that $y^4 - 4 = 0$. ~~T~~ F

F ↑ $2 \mid (-2)$ ↑ doesn't matter if F or T

$\{1, 2\} - \{1\} = \{2\}$

**Faculty information**

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Mariam Reda 75318





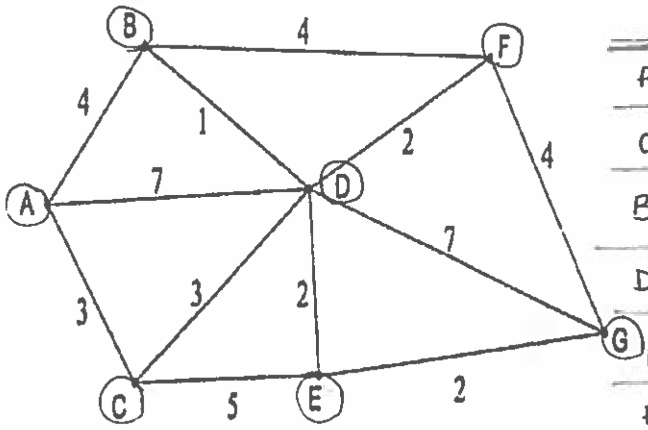
# Final EXAM : MTH 213, ~~Spring~~ Summer 2018

Ayman Badawi

*Score* = $\frac{60}{63}$

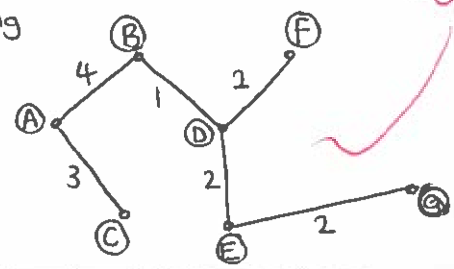
Mariam Reda

**QUESTION 1. (7 points)**
Use Dijkstra's method to find the minimum spanning tree of the below graph.

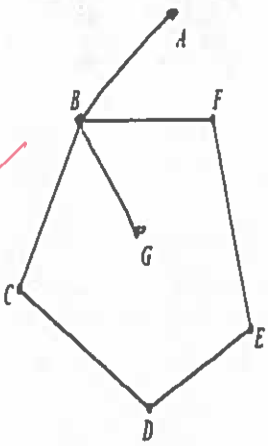


| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | $\boxed{0}$ | $4^A$ | $3^A$ | $7^A$ | ∞ | ∞ | ∞ |
| C | | ∞ $4^A$ | $\boxed{3^A}$ | 3+3 $6^C$ | 3+5 $8^C$ | ∞ | ∞ |
| B | | $\boxed{4^A}$ | | 4+1 $5^B$ | ∞ $8^C$ | 4+4 $8^B$ | ∞ |
| D | | | | $\boxed{5^B}$ | 2+5 $7^D$ | 2+5 $7^D$ | 7+5 $12^D$ |
| E | | | | | $\boxed{7^D}$ | ∞ $7^D$ | 2+7 $9^E$ |
| F | | | | | | $\boxed{7^D}$ | 4+7 $9^E$ |
| G | | | | | | | $\boxed{9^E}$ |

∴ Minimum spanning tree =



**QUESTION 2. (3 points)**



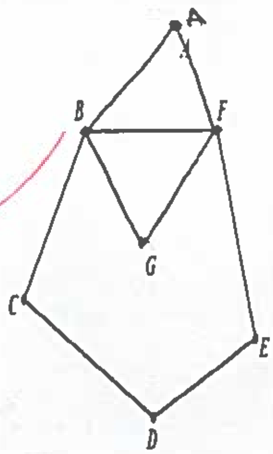
Stare at the graph. You are allowed to add only EDGES (no vertices) so that the graph becomes connected and an Euler Path. that is not Eulerian
You may show the work on the graph itself.

Euler Path ⟶ from starting vertex v, can visit each edge exactly once and end at different vertex y.

⇒ Exactly 2 vertices of odd degrees.
[deg(A) = deg(G) = 1. Rest are even]

[Path example : A – B – F – E – D – C – B – G].

**QUESTION 3. (3 points)**



Stare at the graph. You are allowed to add only EDGES (no vertices) so that the graph becomes connected and Eulerian ( Euler circuit).
You may show the work on the graph itself.

Eulerian ⟶ from starting vertex v, can visit each edge exactly once and return to starting vertex.

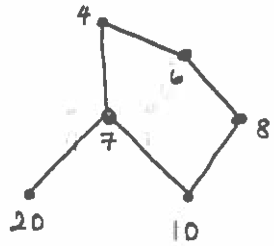
⇒ All vertices must be of even degrees.

[Circuit example : A – B – F – E – D – C – B – G – F – A].

**QUESTION 4.** $A = \{4, 6, 7, 8, 10, 20\}$. Define $\leq$ on $A$ such that $\forall a, b \in A$, $a \leq b$ if and only if $b - a \in \{0, -2, -3, -4, -6, -13, -16\}$. Then $(A, \leq)$ is a partially ordered set (DO NO SHOW THAT)

(i) **(4 points)** Draw the Hassee diagram of such relation

- 4 "≤" : (4).
- 6 "≤" : 4, (6).
- 7 "≤" : 4, (7).
- 8 "≤" : 4, 6, (8).
- 10 "≤" : 4, 6, ~~7~~, 8, (10).
- 20 "≤" : 4, 7, (20).




(ii) **(3 points)** By staring at the Hassee diagram, if possible, find

a. $20 \vee 10$ = 7.

b. $7 \wedge 6$ = 10.

c. $20 \vee 8$ = 4.

d. $20 \vee 4$ = 4.

e. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find $c$ ⇒ Yes, c = 4.

f. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find $m$ ⇒ No, DNE.

**QUESTION 5. (3 points)** Convince me that $|(-\infty, 0]| = |(-5, 4]|$ by constructing a bijective function between the two sets.

Let $f : (-\infty, 0] \longrightarrow (-5, 4]$

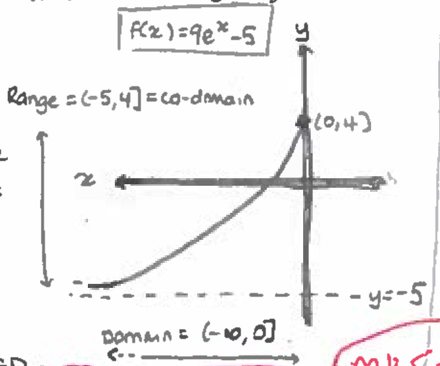
$\therefore f(x) = 9e^x - 5$.

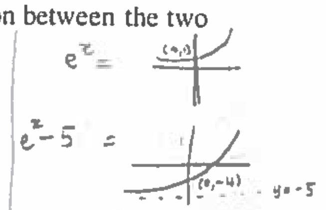
From graph, it is clear that f(x) is one-to-one (since function is increasing) and onto (since range = co-domain).

Hence f is a bijective function.

We know from fact that, if a bijective function can be built, the domain & codomain have same cardinality.

$\therefore |(-\infty, 0]| = |(-5, 4]|$ QED.




$me^0 - 5 = 4$

$m - 5 = 4 \Rightarrow m = 9$.

**QUESTION 6. (6 points)**

(i) Let $F$ be a set with 4 elements, Consider the power set of A, $P(A)$, and let $H = \{d \subset F \mid |d| = 3\}$. Find $|H|$ (i.e., find the cardinality of $H$)

my fault

irrelevant

$|F| = 4$.

H is a set containing all subsets of F for which cardinality of subset = 3.

$\therefore |H|$ = # of all possible subsets d, where $|d| = 3$. (order in subsets not important)

Hence $|H| = \underline{4C3} = 4$ ~~6~~

my fault

supposed to be $P(A)$, so $16C3$

(ii) How many 4-digit even integers greater than 5300 can be formed using the digits (2, 3, 4, 5, 6, 7, 8) such that the third digit must be an even integer.

$1^{st} = 4C1$ (>5000)

$2^{nd} = 6C1$ (>300)

$3^{rd} = 4C1$ (even; {2, 4, 6, 8})

$4^{th} = 4C1$ (even integer).

(-1) 1×6×4×4 + 3×7×4×4

$\therefore$ Total possibilities $= 4C1 \times 6C1 \times 4C1 \times 4C1 = \underline{384}$.

(iii) There are 432 balls and there are 9 holes (very deep holes). The holes are labeled A, A, A, B, B, B, C, C, C. 123 balls must be placed in A-holes (i.e., maybe all of them in one A-hole, or in two A-holes or in three A-holes), 200 balls must be placed in B-holes (see my earlier comment), and the remaining balls must be placed in C-Holes (again, see my earlier comment). Then there are at least $n$ balls that are placed in the same hole (such hole could be an A-hole, or a B-hole, or a C-hole). What is the maximum value of $n$?

|Domain A| = 123. |Co-domain A| = 3.

|Domain B| = 200. |Co-domain B| = 3.

|Domain C| = 432 − 323 = 109. |Co-domain C| = 3.

⇒ least balls in A = $\lceil \frac{123}{3} \rceil = 41$.

least balls in B = $\lceil \frac{200}{3} \rceil = 67$.

least balls in C = $\lceil \frac{109}{3} \rceil = 37$.

$\therefore$ least balls placed in same hole of any kind = min[41, 67, 37]

$\therefore \boxed{n = 37}$.

Answer 67

**QUESTION 7. (2 points)** Find $8^{24002}$ (mod 35)

$\boxed{\text{Fact:}}$ If gcd(a,n) = 1, then $a^{\phi(n)} (\text{mod } n) = 1$.

$n = 35 = 5 \times 7$.

$\phi(n) = 4 \times 6 = 24$.

gcd(8, 35) = 1.

$\therefore 8^{24} (\text{mod } 35) = 1$.

24002 = 24000 + 2

= 1000(24) + 2

$\therefore 8^{24002} (\text{mod } 35) = 8^{24000+2} (\text{mod } 35)$

$= 8^{1000(24)} \cdot 8^2 (\text{mod } 35)$

$= 8^{1000(24)} (\text{mod } 35) \times 8^2 (\text{mod } 35)$

$= 1 \times 64 (\text{mod } 35)$

$= \underline{29}$.

$\therefore 8^{24002} (\text{mod } 35) = \underline{29}$.

**QUESTION 8. (3 points)** Summer

Given $S = \{1,2,3,4,5,6,7,8\}$. Let $f$ be a bijective function from $S$ onto $S$ such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 & 8 \end{pmatrix}$$

(i) Find $f^2$ (i.e., find $f o f$). (Note that by staring at $f$, we understand that $f(1) = 7, \ldots, f(8) = 8$)

$f^2 = f o f = f(f(2))$.

$$\therefore f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 \end{pmatrix}$$

(ii) Find the least positive integer $n$ such that $f^n = I$, where $I$ is the identity map (i.e., $I(a) = a$ for every $a \in S$)

$f$ as disjoint cycle $= \underbrace{(1\ 7)}_{\text{2-cycle}}\underbrace{(2\ 6)}_{\text{2-cycle}}\underbrace{(3\ 5\ 4)}_{\text{3-cycle}}$.

$\therefore n = \mathrm{lcm}[2,2,3] = 6$. $\therefore f^6 = I$.

**QUESTION 9. (3 points)** Let $A = \{0,1,2,3,4\}$ and consider the following equivalence classes of an equivalence relation "=" on A: $[0] = \{0,3,4\}$, $[1] = \{1\}$, $[2] = \{2\}$. Note that the definition of "=" is not given here. Write down the elements (explicitly) of "=" as a subset of $A \times A$.

$[0] = \{0,3,4\}$.
$[1] = \{1\}$
$[2] = \{2\}$.

$\therefore$ Elements of "=" as subset of $A \times A$
$= (0,0), (0,3), (0,4), (3,0), (3,3), (3,4), (4,0), (4,3), (4,4), (1,1), (2,2)$.

**QUESTION 10. (4 points)** Let $A = \{1,2,4,7,8,11,13,14\}$ and let $H = \{1,4,7,13\}$. Define "=" on A such that $\forall a, b \in A$, $a" = "b$ if and only if $ab\ (mod 15) \in H$. Then "=" is an equivalence relation. Do not show that.

(i) Does $3 = 7$? why? $\Rightarrow$ If 3"="7, this means $(3)(7)(\mathrm{mod}\ 15) \in H$. $(3)(7)(\mathrm{mod}\ 15) = 21(\mathrm{mod}\ 15) = 6 \notin H$. Hence, $\boxed{3 \text{"}\neq\text{"} 7}$.

(ii) Does $11 = 14$? why? If 11"="14, then $(11)(14)(\mathrm{mod}\ 15) \in H$. $(11)(14)(\mathrm{mod}\ 15) = 154(\mathrm{mod}\ 15) = 4 \in H$. Hence $\boxed{11 \text{"="} 14}$.

(iii) Find all equivalence classes of $A$

- $[1] = \{1,4,7,13\}$.
- $[2] = \{2,8,11,14\}$.

**QUESTION 11. (4 points)** Let $m = gcd(34, 126)$. Then find $a, b$ such that $m = 34a + 126b$

$126 = 34 \times 3 + 24 \Rightarrow 34 = 24 \times 1 + 10 \Rightarrow 24 = 10 \times 2 + 4 \Rightarrow 10 = 4 \times 2 + 2 \Rightarrow 4 = 2 \times 2 + 0$ (stop)

$\therefore gcd(34, 126) = 2$.

$2 = 10 - 8$
$= 10 - (4 \times 2)$
$= 10 - ((24-20) \times 2)$
$= 10 - (24 \times 2) + (20 \times 2)$
$= (10 \times 1) - (24 \times 2) + (10 \times 4)$
$= (10 \times 5) - (24 \times 2)$
$= ((34-24) \times 5) - (24 \times 2)$
$= (34 \times 5) - (24 \times 5) - (24 \times 2)$
$= (34 \times 5) - (24 \times 7)$
$= (34 \times 5) - ((126-102) \times 7)$
$= (34 \times 5) - (126 \times 7) + (102 \times 7)$
$= (34 \times 5) - (126 \times 7) + ((34 \times 3) \times 7)$
$= (34 \times 5) - (126 \times 7) + (34 \times 21)$
$= (34 \times 26) + (126 \times -7)$

$\therefore$ $\boxed{m = 2.\ a = 26.\ b = -7}$

**QUESTION 12. (4 points)** Use math induction and convince me that $5^{2m}\ (mod\ 20) = 5$ for every $m \geq 1$.

① Prove it for smallest possible m ie. m = 1 :

when m = 1 :

$5^{2m} (\text{mod } 20) = 5^{2(1)} (\text{mod } 20)$

$= 5^2 (\text{mod } 20)$

$= 25 (\text{mod } 20) = 5$ .

② Assume claim is valid for some $m = n$ where $n > 1$ :

ie. assume $5^{2n} (\text{mod } 20) = 5$ .

③ Prove it for (n+1) :

$5^{2(n+1)} (\text{mod } 20) = 5^{2n+2} (\text{mod } 20)$

$= 5^{2n} \cdot 5^{2} (\text{mod } 20)$

$= 5^{2n} (\text{mod } 20) \times 5^2 (\text{mod } 20)$ .

$\Downarrow$ ② $\qquad \Downarrow$ ①

$= (5 \times 5)(\text{mod } 20)$

$= 25 (\text{mod } 20) = 5$ .

Hence $5^{2m} (\text{mod } 20) = 5 \ \forall\, m \geq 1$ .

**QUESTION 13. (4 points)** Let $X$ be number of students in MTH 213. Given $X$ lives in PLANET $Z_{70}$ such that $X\ (mod\ 7) = 2$ and $2X\ (mod 10) = 6$. Use the Chinese remainder Theorem (CRT) and find all possible values of $X$.

X lives in planet $Z_{70} \Rightarrow 0 \leq X < 70$ .

Given : $X (\text{mod } 7) = 2$

$2X (\text{mod } 10) = 6$ .

* Solve $2X (\text{mod } 10) = 6$ over $Z_{10}$ :

$2X (\text{mod } 10) = 6$ has 2 solutions over $Z_{10}$ (since $\gcd(2,6) = 2$ & $2 \mid 10$) .

∴ $x = 3$ & $x = 8$ .

* Check if CRT applies :

$\gcd(7, 10) = 1$ .

Hence, CRT is applicable .

* CRT :

$n = n_1 n_2 = (7)(10) = 70$ .

∘ $m_1 = \frac{n}{n_1} = \frac{70}{7} = 10$ .

$10 x_1 (\text{mod } 7) = 1$ .

∴ $\boxed{x_1 = 5}$

∘ $m_2 = \frac{n}{n_2} = \frac{70}{10} = 7$ .

$7 x_2 (\text{mod } 10) = 1$ .

∴ $\boxed{x_2 = 3}$

we know $0 \leq x < n$ .

since given eqn has 2X , we have two possible solutions in planet $Z_{70}$

∘ when $r_2 = 3$

∘ when $r_2 = 8$ .

$X = (m_1 x_1 r_1 + m_2 x_2 r_2)(\text{mod } n)$ .

∘ $X_1 = \big((10)(5)(2) + (7)(3)(3)\big)(\text{mod } 70)$

$= 163 (\text{mod } 70)$

$= 23$ .

∘ $X_2 = \big((10)(5)(2) + (7)(3)(8)\big)(\text{mod } 70)$

$= 268 (\text{mod } 70)$

$= 58$ .

∴ $X = \{23, 58\}$ .

**QUESTION 14. (6 points)** JUST WRITE T OR F

(i) $\exists! x \in Z$ such that $x^2 - 6x + 9 = 0$. $\longrightarrow$ T.

(ii) $\{5, \{5\}\} - \{5\} = \{5\}$ $\longrightarrow$ F (should be $\{\{5\}\}$) .

(iii) $\forall x \in R, \exists y \in Q^*$ such that $xy \in Q$. $\longrightarrow$ F.

(iv) If $x^2 = 2$ for some $x \in N$, then $xy = \pi\ \forall y \in N$. $\longrightarrow$ F.

(v) If $-7(mod\ 10) = 23(mod\ 10)$, then $-3(mod\ 5) = 23\ (mod\ 5)$ $\longrightarrow$ F.

(vi) $\{3, \{3\}\} \in P(A)$, where $A = \{3, \{3\}, \phi\}$. $\longrightarrow$ T.

-1

**QUESTION 15. (4 points)** Is the sequence 3, 2, 2, 1, 1, 1 graphical? If yes draw such graph.

$S = 3, 2, 2, 1, 1, 1$ .

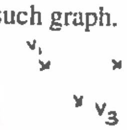
∴ $S' = 1, 1, 0, 1, 1$

$= 1, 1, 1, 1, 0$ .

$S'' = 0, 1, 1, 0$

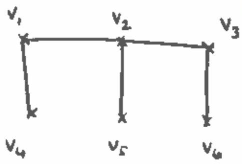
$= 1, 1, 0, 0$ .

$S''' = 0, 0, 0$ . $\Longrightarrow S''' =$




Since $S'''$ is graphical, this means original sequence $S$ is also graphical .

∴ $S =$




$\deg(v_2) = 3$

$\deg(v_1) = 2$

$\deg(v_3) = 2$

$\deg(v_4) = 1$

$\deg(v_5) = 1$

$\deg(v_6) = 1$ .

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